Unit 1

Integer Representation

Skills & Outcomes

- You should master (understand + apply)
  - **Unsigned** binary number to and from decimal
  - **Combinations** that can be made with *n* bits
  - **2's complement** binary to and from decimal
  - Bit sequences to and from **hexadecimal**
  - Predict the outcome & perform **casting**

0’s and 1’s

DIGITAL REPRESENTATION

Information Representation

- Information in computer systems is represented as bits
  - bit = (binary digit) = 0 or 1
- A single bit can only represent 2 values
  To represent more options we use sequences of bits
  - Common sequences:
    8-bits (aka a "byte"), 16-bit, 32-bit and 64-bits
- Kinds of information
  - Numbers, text, code/instructions, sound, images/videos

Interpreting Binary Strings

- Given a sequence of 1's and 0's, you need to know the **representation system** being used, before you can understand the value of those 1's and 0's.
- **Information (value) = Bits + Context (System)**

01000001 = ?

Unsigned Binary system

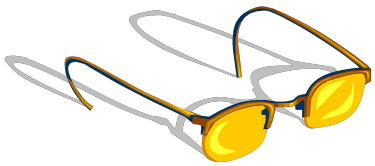
65 decimal

x86 Assembly Instruction

`inc %ecx`

(Add 1 to the ecx register)

ASCII system



'A'$_{ASCII}$

Binary Representation Systems

- **Integer Systems**
  - Unsigned
    - Unsigned binary
  - Signed
    - Signed magnitude
    - 2's complement
    - *Excess-N**
    - *1's complement**
- **Floating Point**
  - For very large and small (fractional) numbers

- **Codes**
  - Text
    - ASCII / Unicode
      $(01000001)_2 = (65)_{10}$ = "A"
  - Decimal Codes*
    - BCD (Binary Coded Decimal) 8421 code:
      $(0100)_2(0001)_2 = (4)_{10}(1)_{10} = (41)_{10}$

* = Not covered in this class

Data Representation

- In C/C++ variables can be of different types and sizes
  - Integer Types on 32-bit (64-bit) architectures

C Type (Signed)	C Type (Unsigned)	Bytes	Bits	x86 Name
char	unsigned char	1	8	byte
short	unsigned short	2	16	word
int / int32_t †	unsigned / uint32_t †	4	32	double word
long	unsigned long	4 (8)	32 (64)	double (quad) word
long long / int64_t †	unsigned long long / uint64_t †	8	64	quad word
char *	-	4 (8)	32 (64)	double (quad) word
int *	-	4 (8)	32 (64)	double (quad) word

  - Floating Point Types

† = defined in stdint.h

C Type	Bytes	Bits	x86 Name
float	4	32	single
double	8	64	double

Using power-of-2 place values

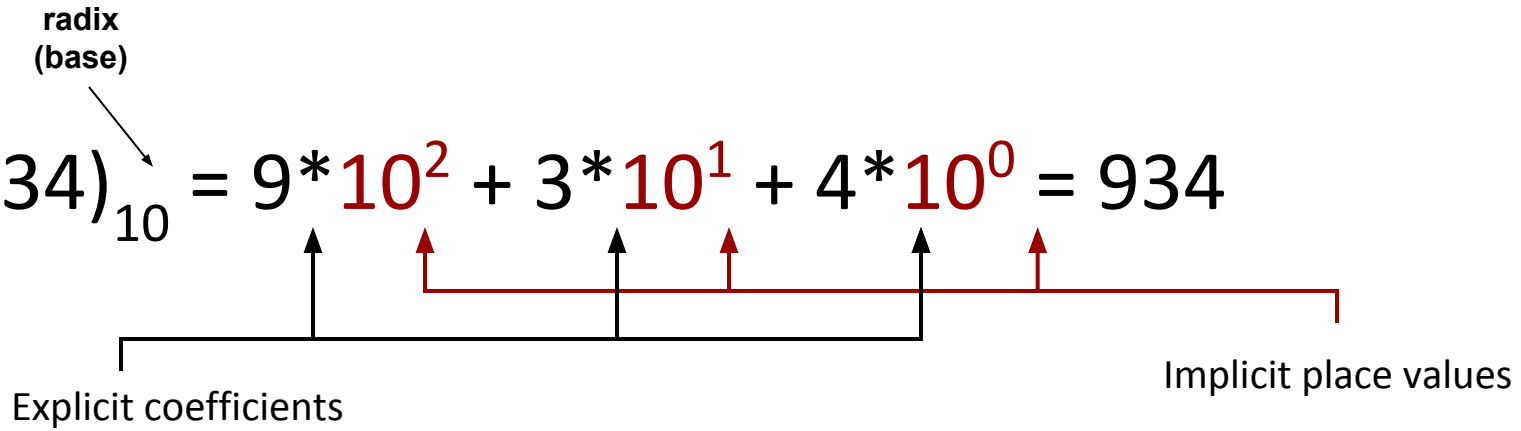
UNSIGNED BINARY TO DECIMAL

Number Systems

- Unsigned binary follows the rules of **positional number systems**
- A positional number systems consist of
  1. a base (radix) *r*
  2. *r* coefficients [0 to *r*-1]
- **Humans:** Decimal (base 10): 0,1,2,3,4,5,6,7,8,9
- **Computers:** Binary (base 2): 0,1
- **Humans working with computers:**
  - Octal (base 8): 0,1,2,3,4,5,6,7
  - Hexadecimal (base 16): 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F
    (digits A through F encode values **10 through 15**)

Anatomy of a Decimal Number

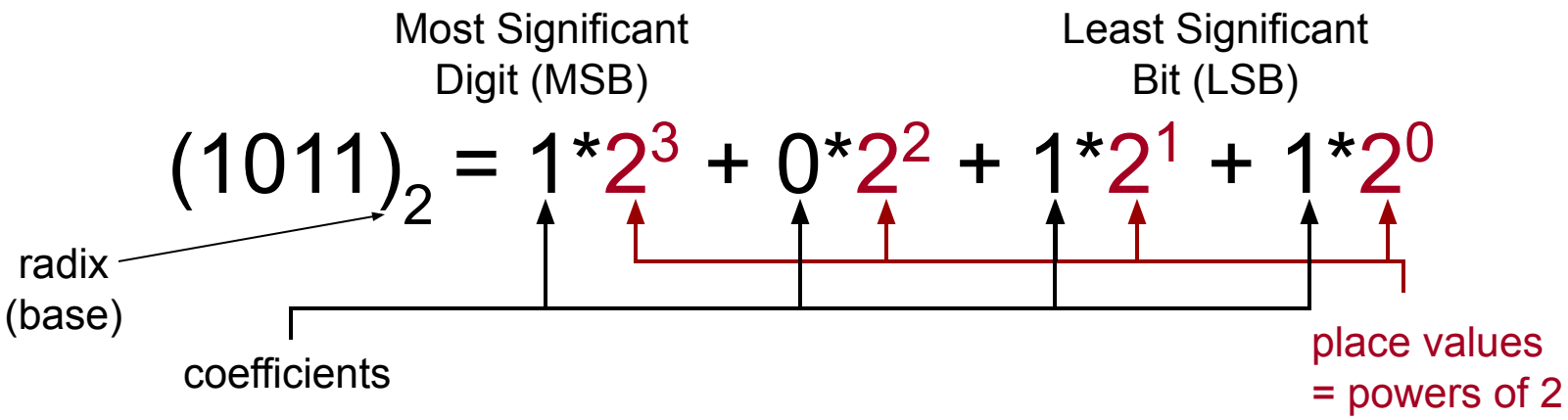
- A number consists of a string of explicit coefficients (digits).
- Each coefficient is multiplied by an implicit place value r^n
- The value of a decimal number (string of decimal coefficients) is the **sum of each coefficient times its place value**



$$(3.52)_{10} = 3*10^0 + 5*10^{-1} + 2*10^{-2} = 3.52$$

Anatomy of an Unsigned Binary Number

- Same as decimal but now the coefficients are 1 and 0 and the place values are the powers of 2



Binary Examples

$$(\underset{8}{1}\underset{4}{0}\underset{2}{0}\underset{1}{1}.\underset{.5}{1})_2 = 8 + 1 + 0.5 = 9.5_{10}$$

$$(\underset{128}{1}\underset{32}{0}\underset{16}{1}0001\underset{1}{)}_2 = 128 + 32 + 16 + 1 = 177_{10}$$

Powers of 2 (memorize these!)

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$
$2^9 = 512$
$2^{10} = 1024$

1024	512	256	128	64	32	16	8	4	2	1

General conversion from unsigned base-*r* to decimal

- An unsigned number in base *r* has place values/weights that are the powers of *r*
- Denote the coefficients as: a_i

$$(a_3 a_2 a_1 a_0 . a_{-1} a_{-2})_r = a_3 * r^3 + a_2 * r^2 + a_1 * r^1 + a_0 * r^0 + a_{-1} * r^{-1} + a_{-2} * r^{-2}$$

Left-most digit = Most Significant Digit (MSD)

Right-most digit = Least Significant Digit (LSD)

$$N_r \Rightarrow \Sigma_i (a_i * r^i) \Rightarrow D_{10}$$

Number in base r

Decimal Equivalent

Examples base-8 and base-16

$$(746)_8 = 7*8^2 + 4*8^1 + 6*8^0$$
$$= 448 + 32 + 6 = 486_{10}$$

$$(1A5)_{16} = 1*16^2 + 10*16^1 + 5*16^0$$
$$= 256 + 160 + 5 = 421_{10}$$

$$(AD2)_{16} =$$

Subtracting powers of 2

UNSIGNED DECIMAL TO BINARY

Decimal to Unsigned Binary

- To convert a decimal number, ***x***, to binary:
  - Only coefficients are 0 and 1. Simply find place values that **add up to the desired values**, starting with larger place values and proceeding to smaller values
  - Place a 1 in those place values and 0 in all others

25 - 16 = 9 9-8 = 1

25_{10} =	0	1	1	0	0	1
	32	16	8	4	2	1

For 25_{10} the place value 32 is too large to include so we include 16. Including 16 means we have to make 9 left over. Include 8 and 1.

Decimal to Unsigned Binary

73_{10} =

0	1	0	0	1	0	0	1
128	64	32	16	8	4	2	1

87_{10} =

0	1	0	1	0	1	1	1

145_{10} =

1	0	0	1	0	0	0	1

0.625_{10} = .

1	0	1	0	0
.5	.25	.125	.0625	.03125

General Algorithm: Integer Part

Binary digits **right-to-left**: divide by 2, take rest

```
87 / 2 = 43 r 1
43 / 2 = 21 r 1
21 / 2 = 10 r 1
10 / 2 =  5 r 0
 5 / 2 =  2 r 1
 2 / 2 =  1 r 0
 1 / 2 =  0 r 1
```

So, $(87)_{10} = (1010111)_2$

General Algorithm: Fraction

Digits **left-to-right**: multiply by 2, take integer part

```
0.1 * 2 = 0.2
0.2 * 2 = 0.4
0.4 * 2 = 0.8
0.8 * 2 = 1.6
0.6 * 2 = 1.2
```

So, $(0.1)_{10} = (0.0\ 0011\ 0011\ \ldots)_2 = 0.00011$

Decimal to Another Base

- To convert a decimal number, *x,* to base *r*:
  - Use the place values of base *r* (powers of *r*). Starting with largest place values, fill in **coefficients** that, multiplied by the implicit place value, sum up to desired decimal value.

$$75_{10} = \underset{256}{0} \quad \underset{16}{4} \quad \underset{1}{B} \quad \text{hex}$$

The 2^n rule

UNIQUE COMBINATIONS

Unique Combinations

- Given *n* digits of base *r*, how many unique numbers can be formed? **r^n**
  - What is the range? **[0 to r^n-1]**

2-digit, decimal numbers (r=10, n=2) — **100 combinations: 00-99**

					0-9	0-9

3-digit, decimal numbers (r=10, n=3) — **1000 combinations: 000-999**

				___	___	___

4-bit, binary numbers (r=2, n=4) — **16 combinations: 0000-1111**

			0-1	0-1	0-1	0-1

6-bit, binary numbers (r=2, n=6) — **64 combinations: 000000-111111**

___	___	___	___	___	___	___

Main Point: Given n digits of base r, **r^n** unique numbers can be made with the range **0** to **r^n-1**

Corollary: Unsigned with all 1's

- What is the decimal value of an unsigned **binary string of all 1's**?
  - "111" is 2^3-1 = 7 (4 + 2 + 1)
  - "1111" is 2^4-1 = 15 (8 + 4 + 2 + 1)
  - "11111111" is 2^8-1 = 255
  - "11111111 11111111" is 2^{16}-1 = 65535
- What is the decimal value of an unsigned **hex string of all F's**?
  - "FF" is 16^2-1 = $(2^4)^2$ - 1 = 2^8 - 1 = 255
  - "FFFF" is 16^4-1 = $(2^4)^4$ - 1 = 2^{16} - 1 = 65535

Range of C data types

- For an unsigned integer data type: **[0, 2^n-1]**
- For a signed integer data type: **[-2^{n-1}, 2^{n-1}-1]**
  - We use half combinations for negative and half for positive (0 is considered a positive number by common integer convention)

Bytes	Bits n	Type	Unsigned Range	Signed Range
1	8	[unsigned] char	**0 to 255**	**-128 to +127**
2	16	[unsigned] short	**0 to 65535**	**-32768 to +32767**
4	32	[unsigned] int	0 to 4,294,967,295 **(0 to +4 billion)**	-2,147,483,648 to +2,147,483,647 **(-2 billion to +2 billion)**
8	64	[unsigned] long	0 to 18,446,744,073,709,551,615 **(0 to +18 billion billion)**	-9,223,372,036,854,775,808 to +9,223,372,036,854,775,807 **(-9 to +9 billion billion)**
4 (8)	32 (64)	char *	0 to 4,294,967,295 0 to 18,446,744,073,709,551,615	

- How will I ever remember those ranges?

Approximating Large Powers of 2

- We often need to find the decimal approximation of a large power of 2 like 2^{16}, 2^{32}, etc.
- Use following approximations:
  - 1 kilo- is $2^{10} \approx 10^3$ (1 thousand)
  - 1 Mega- is $2^{20} \approx 10^6$ (1 million)
  - 1 Giga- is $2^{30} \approx 10^9$ (1 billion)
  - 1 Tera- is $2^{40} \approx 10^{12}$ (1 trillion)
- For other powers of 2, decompose into product of 2^{10} or 2^{20} or 2^{30} and a power of 2 that is less than 2^{10}
  - 16-bit word: 64k numbers
  - 32-bit dword: 4G numbers
  - 64-bit qword: 16 million trillion numbers (or 16 billion billion numbers)

$$2^{16} = 2^6 * 2^{10} \approx 64 * 10^3 = 64{,}000$$

$$2^{24} = 2^4 * 2^{20} \approx 16 * 10^6 = 16{,}000{,}000$$

$$2^{28} = 2^8 * 2^{20} \approx 256 * 10^6 = 256{,}000{,}000$$

$$2^{32} = 2^2 * 2^{30} \approx 4 * 10^9 = 4{,}000{,}000{,}000$$

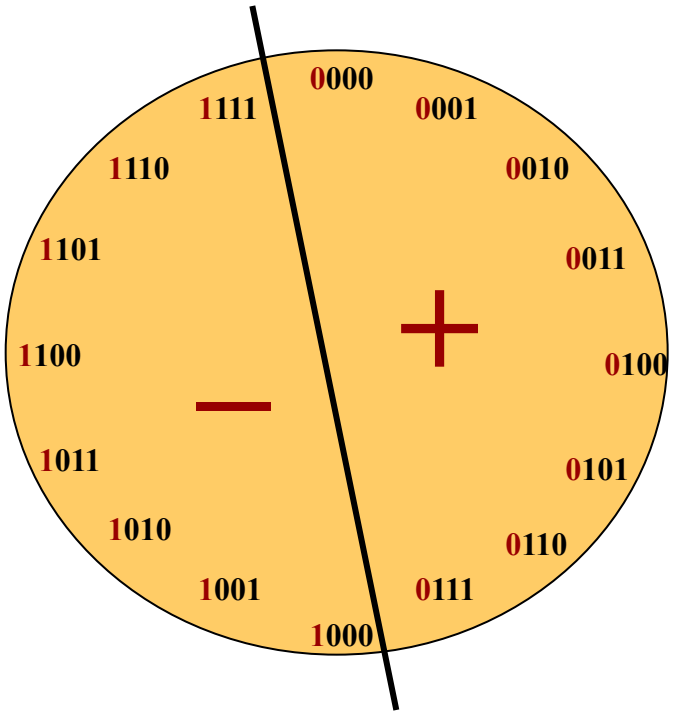
$$2^{64} = 2^4 * 2^{60} = 2^4 * 2^{20} * 2^{40} \approx 16 * 10^6 * 10^{12} = 2^4 * 2^{30} * 2^{30} \approx 16 * 10^9 * 10^9$$

2’s complement

SIGNED NUMBERS TO DECIMAL

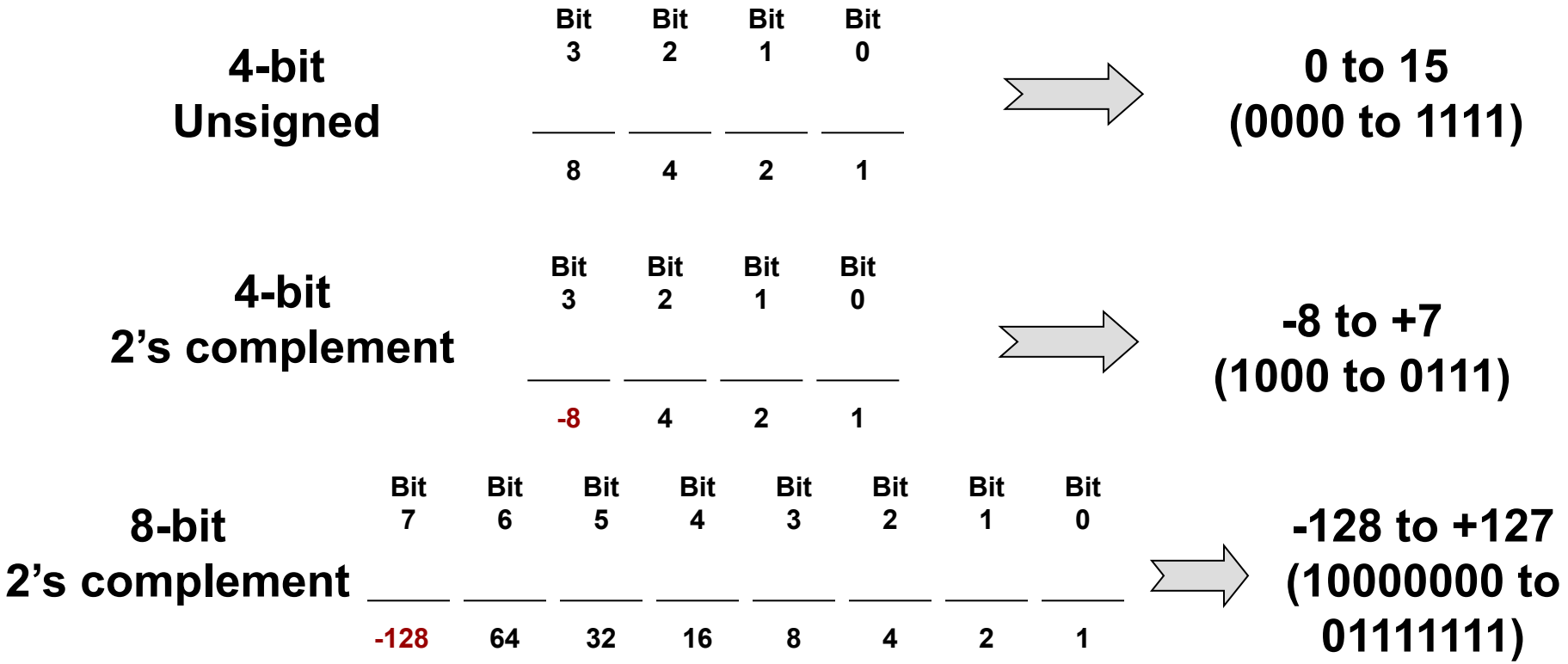
Signed numbers

- Systems used to represent signed numbers split binary combinations in half
  - half for positive, incl. zero
  - half for negative
- Generally, positive and negative numbers are distinguished by the MSB
  - MSB=1 means negative
  - MSB=0 means positive



2's Complement System

- Only change to place values: negative value of MSB
  - MSB of 1 has value of -2^{n-1}



2's Complement Examples

4-bit 2's complement

1	0	1	1	= -5
-8	4	2	1	
0	0	1	1	= +3
-8	4	2	1	
1	1	1	1	= -1
-8	4	2	1	

Notice that +3 in 2's comp. is the **same as in the unsigned system**

8-bit 2's complement

1	0	0	0	0	0	0	1	= -127
-128	64	32	16	8	4	2	1	
0	0	0	1	1	0	0	1	= +25
-128	64	32	16	8	4	2	1	

Important: Positive numbers have the same representation in 2's complement as in normal unsigned binary

2's Complement Range

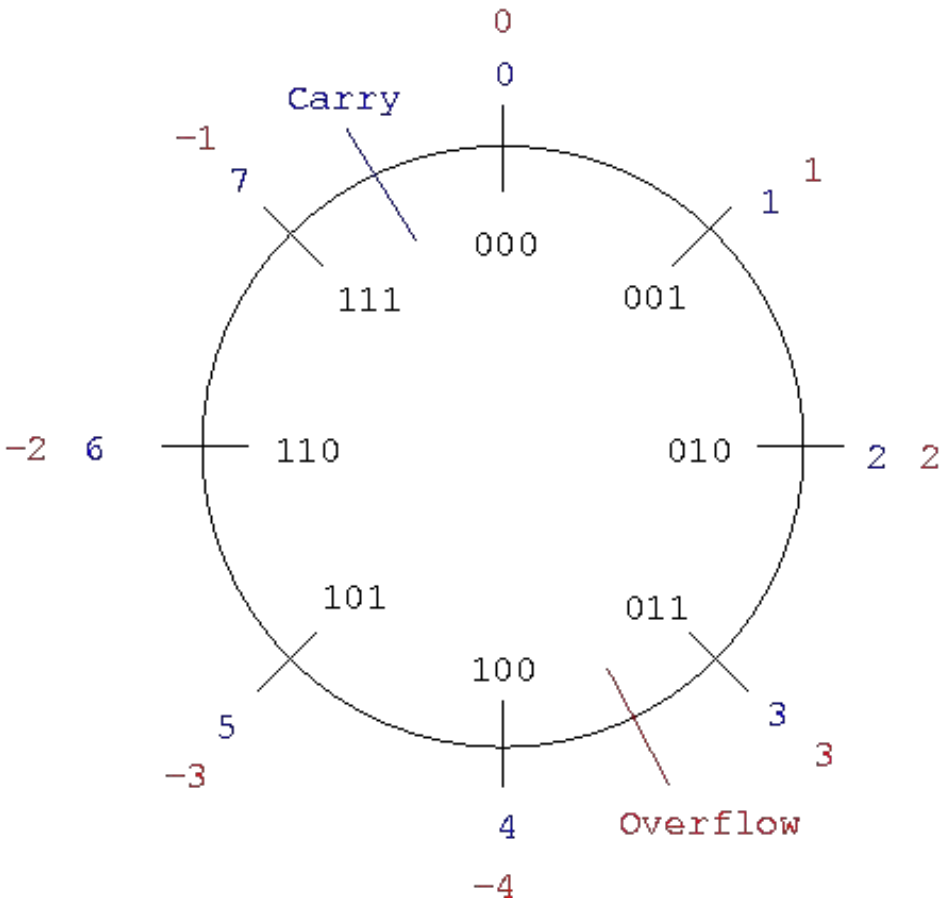
- Given n bits…
  - Max positive value = 011…11
    - Includes all n-1 positive place values
  - Max negative value = 100…00
    - Includes only the negative MSB place value

Range with n bits of 2's complement

[-2^{n-1} to $2^{n-1}-1$]

- $(0)_{10}$ always encoded as 000…00
- $(-1)_{10}$ always encoded as 111…11
- $(1)_{10}$ always encoded as 000…01

Wheel of Integers



Unsigned and Signed Variables

- In C, unsigned variables use unsigned binary (pos. power-of-2 place values) to represent numbers

1	0	0	1	0	0	1	1	= +147
128	64	32	16	8	4	2	1	

- In C, signed variables use the 2's complement system (neg. MSB weight) to represent numbers

1	0	0	1	0	0	1	1	= -109
-128	64	32	16	8	4	2	1	

IMPORTANT NOTE

- All computer systems use the **2's complement system** to represent **signed integers**!
- So from now on, if we say an integer is **signed**, we are actually saying it uses the 2's complement system unless otherwise specified
  - Other systems like "signed magnitude" or "1's complement" exist but will not be used (signed multitude: sign bit + value, so -1 is 100..01)

Zero and Sign Extension

- Extension is the process of increasing the number of bits used to represent a number **without changing its value**

Unsigned = Zero Extension (**always add leading 0's**):

111011 = **00111011**

Increase a 6-bit number to 8-bit number by zero extending

2's complement = Sign Extension (**replicate sign bit**):

positive **011010** = **00011010**

negative **110011** = **11110011**

Sign bit is just repeated as many times as necessary

Why does it work?

111… = -128 + 64 + 32 = -32 and 1… = -32

Zero and Sign Truncation

- Truncation is the process of decreasing the number of bits used to represent a number **without changing its value**

Unsigned = Zero Truncation (Remove leading 0's):

~~00~~111011 = 111011

Decrease an 8-bit number to 6-bit number by truncating 0's. Can't remove a '1' because value is changed

2's complement = Sign Truncation (Remove copies of sign bit):

positive ~~00~~011010 = 011010

negative ~~111~~10011 = 10011

Any **copies** of the MSB can be removed without changing the numbers value. **Be careful not to change the sign by cutting off ALL the sign bits.**

Shortcuts for Converting Binary to Hexadecimal

SHORTHAND FOR BINARY

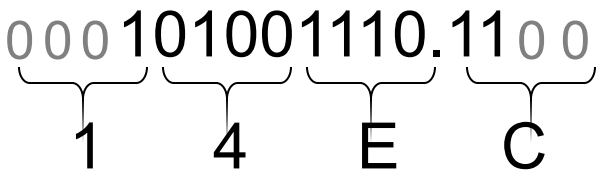
Binary and Hexadecimal

- Hex is base 16 which is 2^4
- 1 Hex digit $(?)_{16}$ can represent: 0-F $(0\text{-}15)_{10}$
- 4 bits of binary $(? ? ? ?)_2$ can represent: $0000\text{-}1111 = 0\text{-}15_{10}$
- Conclusion…

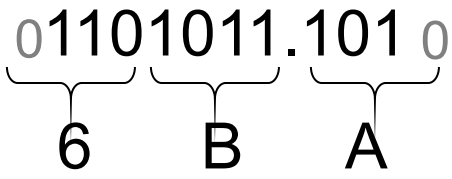
  1 hex digit = 4 bits

Binary to Hex

- Make groups of 4 bits starting from radix point and working outward
- Add leading or trailing 0's where necessary
- Convert each group of 4 to an hex digit



$= 14E.C_{16}$



$= 6B.A_{16}$

Hex to Binary

- Expand each hex digit to a group of 4 bits

$14E.C_{16}$

$0\,0\,0\,1\,0\,1\,0\,0\,1\,1\,1\,0.1\,1\,0\,0$

$= 101001110.11_2$

$D93.8_{16}$

$1\,1\,0\,1\,1\,0\,0\,1\,0\,0\,1\,1.1\,0\,0\,0_2$

$= 110110010011.1_2$

(leading or trailing 0's can be discarded)

Use of Hex Representation

- Values in modern computers use many bits!
- We use hexadecimal as a shorthand notation (4 bits = 1 hex digit)
  - 1101 0010 = D2 hex or **0xD2** if you write it in C/C++
  - 0111 0110 1100 1011 = 76CB hex or **0x76CB** in C/C++

Interpreting Hex Strings

- What does the following hexadecimal represent?
- Just like binary, you must know the underlying *representation system* being used before you can interpret a hex value
- Information (value) = Hex + Context (System)

0x41 = ?

Unsigned Binary system

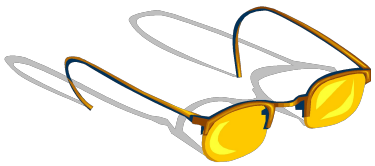
65 decimal

x86 Assembly Instruction

`inc %ecx`

(Add 1 to the ecx register)

ASCII system



'A'$_{ASCII}$

Hexadecimal & Sign

- If a number is represented in 2's complement (e.g. 10010110) then the MSB of its binary representation would correspond to:
  - 0 for positive numbers (incl. 0)
  - 1 for negative numbers
- If that same 2's complement number is viewed as hex (e.g. 0x96), how can we tell if its value is positive or negative?
  - **MSD of 0-7 = Positive**
  - **MSD of 8-F = Negative**

Hex – Binary – Sign
0 = 0000 = Pos
1 = 0001 = Pos
2 = 0010 = Pos
3 = 0011 = Pos
4 = 0100 = Pos
5 = 0101 = Pos
6 = 0110 = Pos
7 = 0111 = Pos
8 = 1000 = Neg
9 = 1001 = Neg
A = 1010 = Neg
B = 1011 = Neg
C = 1100 = Neg
D = 1101 = Neg
E = 1110 = Neg
F = 1111 = Neg

Implicit and Explicit

APPLICATION: CASTING

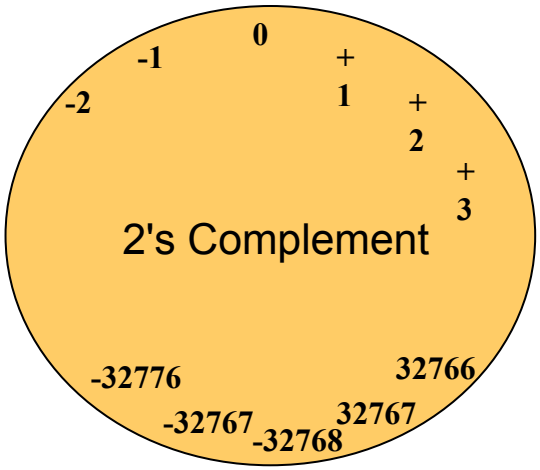
Implicit and Explicit Casting

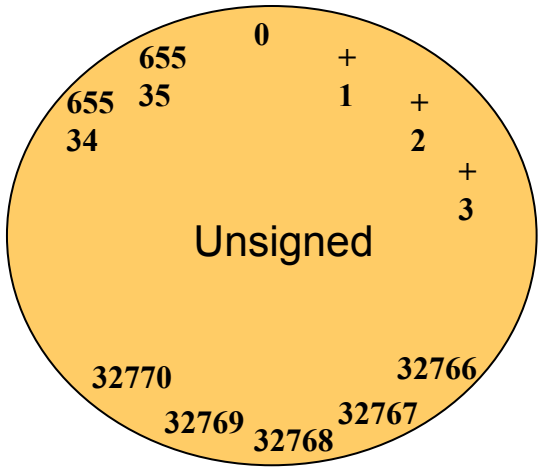
- Use your understanding of unsigned and 2's complement to predict the output
- Notes:
  - unsigned short range: 0 to 65535
  - signed short range: -32768 to +32767

```
int main()
{
  short int v = -10000; /* 0xd8f0 */
  unsigned short uv = (unsigned short) v;
  printf("v = %d, uv = %u\n", v, uv);
  return 0;
}
```

Expected Output:

```
v = -10000, uv = 55536
```






```
int main()
{
  unsigned u = 4294967295u;  /* UMax */
  int tu = (int) u;
  printf("u = %u, tu = %d\n", u, tu);
  return 0;
}
```

Expected Output:

```
u = 4294967295, tu = -1
```

Implicit and Explicit Casting

- Use your understanding of zero and sign extension to predict the output

```
int main()
{
  short int v = 0xcfc7;       /* -12345 */
  int vi = v;                  /* ????? */
  unsigned short uv = 0xcfc7; /* 53191 */
  unsigned uvi = uv;           /* ????? */
  printf("vi = %x, uvi = %x\n", vi, uvi);
  return 0;
}
```

Expected Output:

```
vi = ffffcfc7, uvi = cfc7
```

```
int main()
{
  int x = 53191;              /* 0xcfc7 */
  short sx = x;
  int y = sx;
  char z = x;

  printf("sx = %d, y = %d ", sx, y);
  printf("z = %d\n", z);
  return 0;
}
```

Expected Output:

```
sx = -12345, y = -12345, z = -57
```

Advice

- Casting can be done implicitly and explicitly
- Casting from one system to another applies a new "interpretation" (pair of glasses) on the same bits
- Casting from one size to another will perform extension or truncation (based on the system)